Wadge Hierarchy of Omega Context Free Languages

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Abstract

The main result of this paper is that the length of the Wadge hierarchy of omega context free languages is greater than the Cantor ordinal ε_0, and the same result holds for the conciliating Wadge hierarchy, defined in [Dup99], of infinitary context free languages, studied in [Bea84a]. In the course of our proof, we get results on the Wadge hierarchy of iterated counter ω-languages, which we define as an extension of classical (finitary) iterated counter languages to ω-languages.

Keywords: omega context free languages; topological properties; Wadge hierarchy; conciliating Wadge hierarchy; infinitary context free languages; iterated counter ω-languages.

Contents

1 Introduction

Since J.R. Büchi studied the ω-languages recognized by finite automata to prove the decidability of the monadic second order theory of one successor over the integers [Büc60a] the so called ω-regular languages have been intensively studied. See [Tho90] and [PP98] for many results and references.

As Pushdown automata are a natural extension of finite automata, R. S. Cohen and A. Y. Gold [CG77] , [CG78] and M. Linna [Lin76] studied the ω-languages accepted by omega pushdown automata, considering various acceptance conditions for omega words. It turned out that the omega languages accepted by omega pushdown automata were also those generated by context free grammars where infinite derivations are considered , also studied by M. Nivat [Niv77] ,[Niv78] and L. Boasson and M. Nivat [BN80]. These languages were then called the omega context free languages (ω-CFL). See also Staiger's paper [Sta97] for a survey of general theory of ω-languages, including more powerful accepting devices , like Turing machines.

Topological properties of ω-regular languages were first studied by L. H. Landweber in [Lan69] where he showed that these languages are boolean combination of G_δ sets and that one can decide whether a given ω-regular language is in a given Borel class. It turned out that an ω-regular language is in the class $\mathbf{G}_\delta$ iff it is accepted by a deterministic Büchi automaton. These results have been extended to deterministic pushdown automata in [Lin77][Fin99a]. But (non deterministic) omega context free languages exhaust the hierarchy of Borel sets of finite rank and it is undecidable to determine the Borel rank of an ω-CFL [Fin99a].

The hierarchy induced on ω-regular languages by the Borel Hierarchy was refined in [Bar92] and [Kam85] but K. Wagner had found the most refined one, now called the Wagner hierarchy and which is the hierarchy induced on ω-regular languages by the Wadge Hierarchy of Borel sets [Wag79].

This paper is mainly a study of the Wadge hierarchy of context free and iterated counter ω-languages:

We study iterated counter ω-languages which are an extension of the well known iterated counter languages to ω-languages. The class of iterated counter languages is divided into an infinite hierarchy of subclasses of the class of context free languages which can be defined by means of substitution by counter languages or by some restrictions on the pushdown automaton: the words in the pushdown store always belong to a bounded language in the form $(z_k)^\star...(z_2)^\star(z_1)^\star Z_0$, where $\{Z_0, z_1, ..., z_k\}$ is the pushdown alphabet [Ber79][ABB96]. Thus these automata are X-automata in the sense of J. Engelfriet and H. J. Hoogeboom who initiated the study of general storage type for machines reading infinite words [EH93]. The study of topologically defined hierarchies of ω-languages accepted by such X-automata is asked by W. Thomas and H. Lescow [LT94].

To study the Wadge hierarchy of these languages, we shall use results of J. Duparc about the Wadge hierarchy of Borel sets. In [Dup99] [Dup95a] he gave a normal form of Borel sets of finite rank, i.e. an inductive construction of a Borel set of every given degree in the Wadge hierarchy of Borel sets of finite rank. In the course of the proof he studied the concilating hierarchy which is a hierarchy of sets of finite **and** infinite sequences. The conciliating hierarchy is closely related to the Wadge hierarchy of non self dual sets.

On the other hand the infinitary languages, i.e. $(\leq \omega)$-languages (containing finite **and** infinite words), accepted by pushdown automata have been studied in [Bea84a][Bea84b] where D. Beauquier considered these languages as process behaviours which may terminate or not, as for transition systems studied in [AN82]. We continue this study, giving results on the conciliating hierarchy of infinitary iterated counter languages and showing that the length of the conciliating hierarchy of infinitary context free languages is greater than the Cantor ordinal ε_0.

Then we study the Wadge hierarchy of omega context free languages, showing that the length of the Wadge hierarchy of k-iterated counter languages is greater than the ordinal $\omega(k+2)$ obtained by $k+2$ iterations of the operation

of ordinal exponentiation of base ω. More precisely $\omega(0) = 1$ and for each integer $n \geq 0$, $\omega(n+1) = \omega^{\omega(n)}$. We then deduce that the Wadge hierarchy of omega context free languages has length greater than ε_0 which is a much larger ordinal than:
ω^ω which is the length of the hierarchy of ω-regular languages, [Wag79], and $\omega^{(\omega^2)}$ which is the length of the hierarchy of **deterministic** context free ω-languages, [Dup99][Fin99b].

In section 2, we first review some above definitions and results about ω-regular, ω-context free languages, and infinitary context free languages.

In section 3, we extend the definition of (k-) iterated counter (finitary) languages to ω-languages and we show that these latter languages verify some characterizations by means of automata with (k-) iterated counter storage type as well as by means of omega Kleene closure of finitary languages.

In section 4, we recall some basic facts about Borel and Wadge hierarchies and we prove that the Wadge hierarchy of ω-context free languages is non effective.

In section 5, we introduce Duparc's operations on conciliating sets and we investigate closure properties, with regard to these operations, of classes of iterated counter infinitary languages.

In section 6, we apply preceding properties to the study of the conciliating hierarchy of infinitary context free languages.

In section 7, we prove results about the length of the Wadge hierarchies of ω-context free languages and of iterated counter ω-languages.

2 ω-regular and ω-context free languages

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages, see for example [HU69] ,[Tho90]. We first recall some of the definitions and results concerning ω-regular and ω-context free languages and omega pushdown automata as presented in [Tho90] [CG77] , [CG78].
When Σ is a finite alphabet, a finite string (word) over Σ is any sequence $x = x_1 \ldots x_k$, where $x_i \in \Sigma$ for $i = 1, \ldots, k$,and k is an integer ≥ 1. The length of x is k, denoted by $|x|$.

If $|x| = 0$, x is the empty word denoted by λ.
we write $x(i) = x_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$ and $x[0] = \lambda$.
$\Sigma^\star$ is the set of finite words over Σ.
The first infinite ordinal is ω.
An ω-word over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$.
When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$
and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ the finite word of length n, prefix of σ.
The set of ω-words over the alphabet Σ is denoted by Σ^ω.
An ω-language over an alphabet Σ is a subset of Σ^ω.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:
$(u.v)(k) = u(k)$ if $k \leq |u|$, and
$(u.v)(k) = v(k - |u|)$ if $k > |u|$.

For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega / u_i \in V, \forall i \geq 1\}$ is the ω-power of V.
For $V \subseteq \Sigma^\star$, the complement of V (in $\Sigma^\star$) is $\Sigma^\star - V$ denoted V^-.
For a subset $A \subseteq \Sigma^\omega$, the complement of A is $\Sigma^\omega - A$ denoted A^-.
When we consider subsets of $\Sigma^{\leq\omega} = \Sigma^\star \cup \Sigma^\omega$, if $A \subseteq \Sigma^{\leq\omega}$ then $A^- = \Sigma^{\leq\omega} - A$, but when $A = B \cup C$ with $B \subseteq \Sigma^\star$ and $C \subseteq \Sigma^\omega$ we shall use the notation B^- for $\Sigma^\star - B$ and C^- for $\Sigma^\omega - C$ when this will be clear from the context.

The prefix relation is denoted $\sqsubseteq$: the finite word u is a prefix of the finite word v (denoted $u \sqsubseteq v$) if and only if there exists a (finite) word w such that $v = u.w$.
This definition is extended to finite words which are prefixes of ω-words:
the finite word u is a prefix of the ω-word v (denoted $u \sqsubseteq v$) iff there exists an ω-word w such that $v = u.w$.

Definition 2.1 *: A finite state machine (FSM) is a quadruple $M = (K, \Sigma, \delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state and δ is a mapping from $K \times \Sigma$ into 2^K . A FSM is called deterministic (DFSM) iff : $\delta : K \times \Sigma \rightarrow K$.*
A Büchi automaton (BA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a finite state machine and $F \subseteq K$ is the set of final states.
A Muller automaton (MA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a FSM and $F \subseteq 2^K$ is the collection of designated state sets.
A Büchi or Muller automaton is said deterministic if the associated FSM is deterministic.

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ.
A sequence of states $r = q_1 q_2 \ldots q_n \ldots$ is called an (infinite) run of $M = (K, \Sigma, \delta, q_0)$ on σ, starting in state p, iff: 1) $q_1 = p$ and 2) for each $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$.
In case a run r of M on σ starts in state q_0, we call it simply "a run of M on σ ".
For every (infinite) run $r = q_1 q_2 \ldots q_n \ldots$ of M, $In(r)$ is the set of states in K entered by M infinitely many times during run r:
$In(r) = \{q \in K / \{i \geq 1 / q_i = q\}$ is infinite $\}$.
For $M = (K, \Sigma, \delta, q_0, F)$ a BA, the ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega$ / there exists a run r of M on σ such that $In(r) \cap F \neq \emptyset\}$.
For $M = (K, \Sigma, \delta, q_0, F)$ a MA, the ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega$ / there exists a run r of M on σ such that $In(r) \in F\}$.

The classical result of R. Mc Naughton [MaN66] established that the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA (NDMA) which is also equal to the expressive power of non deterministic BA (NDBA) .
There is also a characterization of the languages accepted by MA by means of the "ω-Kleene closure" which we give now the definition:

Definition 2.2 *For any family L of finitary languages over the alphabet Σ, the ω-Kleene closure of L, is :*

$$\omega - KC(L) = \{\cup_{i=1}^n U_i . V_i^\omega / U_i, V_i \in L, \forall i \in [1, n]\}$$

Theorem 2.3 *For any ω-language L, the following conditions are equivalent:*

1. *L belongs to $\omega - KC(REG)$, where REG is the class of (finitary) regular languages.*
2. *There exists a DMA that accepts L.*
3. *There exists a MA that accepts L.*
4. *There exists a BA that accepts L.*

An ω-language L satisfying one of the conditions of the above Theorem is called an ω-regular language (or regular ω-language). The class of ω-regular languages will be denoted by REG_ω.

We now define the pushdown machines and the classes of ω-context free languages.

Definition 2.4 *A pushdown machine (PDM) is a 6-tuple* $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$*, where* K *is a finite set of states,* Σ *is a finite input alphabet,* Γ *is a finite pushdown alphabet,* $q_0 \in K$ *is the initial state,* $Z_0 \in \Gamma$ *is the start symbol, and* δ *is a mapping from* $K \times (\Sigma \cup \{\lambda\}) \times \Gamma$ *to finite subsets of* $K \times \Gamma^\star$ *.*
If $\gamma \in \Gamma^+$ *describes the pushdown store content, the leftmost symbol will be assumed to be on " top" of the store. A configuration of a PDM is a pair* (q, γ) *where* $q \in K$ *and* $\gamma \in \Gamma^\star$*.*
For $a \in \Sigma \cup \{\lambda\}$*,* $\alpha, \gamma \in \Gamma^\star$ *and* $Z \in \Gamma$*, if* (p, β) *is in* $\delta(q, a, Z)$*, then we write* $a : (q, Z\gamma) \mapsto_M (p, \beta\gamma)$*.*
$\mapsto_M^\star$ *is the transitive and reflexive closure of* $\mapsto_M$*. (The subscript* M *will be omitted whenever the meaning remains clear).*

Let $\sigma = a_1 a_2 \ldots a_n$ *be a finite word over* Σ*. A (finite) sequence of configurations* $r = (q_i, \gamma_i)_{1 \leq i \leq m}$ *is called a complete run of* M *on* σ*, starting in configuration* (p, γ)*, iff:*

1. $(q_1, \gamma_1) = (p, \gamma)$
2. *for each* $i \in [1, m-1]$*, there exists* $b_i \in \Sigma \cup \{\lambda\}$ *satisfying* $b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$ *such that* $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_m$

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ *be an* ω*-word over* Σ*. An infinite sequence of configurations* $r = (q_i, \gamma_i)_{i \geq 1}$ *is called a complete run of* M *on* σ*, starting in configuration* (p, γ)*, iff:*

1. $(q_1, \gamma_1) = (p, \gamma)$
2. *for each* $i \geq 1$*, there exists* $b_i \in \Sigma \cup \{\lambda\}$ *satisfying* $b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$ *such that* $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$

As for FSM, for every such run, $In(r)$ *is the set of all states entered infinitely often during run* r*.*
A complete run r *of* M *on* σ *, starting in configuration* (q_0, Z_0)*, will be simply called " a run of* M *on* σ *".*

Definition 2.5 *A Büchi pushdown automaton (BPDA) is a 7-tuple* $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *where* $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ *is a PDM and* $F \subseteq K$ *is the set of final states.*
The ω*-language accepted by* M *is* $L(M) = \{\sigma \in \Sigma^\omega$ */ there exists a complete run* r *of* M *on* σ *such that* $In(r) \cap F \neq \emptyset\}$*.*

Definition 2.6 *A Muller pushdown automaton (MPDA) is a 7-tuple* $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *where* $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ *is a PDM and* $F \subseteq 2^K$

is the collection of designated state sets.
The ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega$ / there exists a complete run r of M on σ such that $In(r) \in F\}$.

Remark 2.7 *We consider here two acceptance conditions for ω-words , the Büchi and the Muller acceptance conditions, respectively denoted 2-acceptance and 3-acceptance in [Lan69] and in [CG78] and $(inf, \sqcap)$ and $(inf, =)$ in [Sta97].*

Remark 2.8 *Without loss of generality we can always assume that the pushdown alphabet is $\Gamma = \{Z_0\} \cup \Gamma'$ where Γ' does not contain the symbol Z_0. And we can assume that the start symbol remains, during any finite or infinite computation, at the bottom of the store, and appears only there, i.e. that the content of the pushdown store is always in the form γZ_0 where $\gamma \in (\Gamma - \{Z_0\})^\star$.*

R.S. Cohen and A.Y. Gold, and independently M. Linna, established a characterization Theorem for ω-CFL:

Theorem 2.9 *Let CF be the class of context free (finitary) languages. Then for any ω-language L the following three conditions are equivalent:*

1. *$L \in \omega - KC(CF)$.*
2. *There exists a $BPDA$ that accepts L.*
3. *There exists a $MPDA$ that accepts L.*

In [CG77] are also studied the ω-languages generated by ω-context free grammars and it is shown that each of the conditions 1), 2), and 3) of the above Theorem is also equivalent to: 4) L is generated by a context free grammar G by leftmost derivations. These grammars are also studied in [Niv77] , [Niv78].
Then we can let the following definition:

Definition 2.10 *An ω-language is an ω-context free language (ω-CFL) (or context free ω-language) iff it satisfies one of the conditions of the above Theorem.*

If finite and infinite words are viewed as process behaviours, it is natural to consider the infinitary languages (containing finite **and** infinite words) recognized by transition systems [AN82]. the infinitary languages accepted by pushdown machines have been studied in [Bea84a], [Bea84b]. A pushdown machine $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is given with subsets K_1 and K_2 of K: K_1 is used for acceptation of finite words by final states (in K_1) and K_2 is used for acceptation of ω-words by a Büchi condition with the set K_2 as set of final states. The set of (finite or infinite) words accepted by the pushdown machine in such a way is the union of a finitary context free language and of an ω-CFL [Bea84a]. Then we let the following:

Definition 2.11 *Let X be a finite alphabet. A subset L of $X^{\leq\omega}$ is said to be an infinitary context free language iff there exist a finitary context free language $L_1 \subseteq X^\star$ and an ω-CFL $L_2 \subseteq X^\omega$ such that $L = L_1 \cup L_2$.*

3 Iterated counter ω-languages

Recall first that a rational cone is a class of (finitary) languages which is closed under morphism, inverse morphism, and intersection with a rational language (or, equivalently to these three properties, closed under rational transduction), [Ber79].

Definition 3.1 ([Lat83]) *Let Rocl (restricted one counter languages) be the family of (finitary) languages accepted by pushdown automata, with a push-down alphabet containing only one symbol which is the start symbol Z_0, by empty storage and accepting states. It is also the rational cone generated by the semi-Dyck language $D_1'^\star$ over one pair of parentheses.*

We consider now a pushdown automaton with a pushdown alphabet in the form $\Gamma = \{Z_0, a\}$ (Z_0 is the bottom symbol as in the remark 2.8 and it always remains at the bottom of the pushdown store). It is called a one counter automaton. The languages accepted by such automata have been much studied. It turned out that these languages are obtained by substituting languages of *Rocl* in languages of *REG*:

Definition 3.2 ([Lat83][Ber79]) *Let OCL be the family of (finitary) languages accepted by one counter automata by final states.*

Recall now the definition of substitution in languages: A substitution f is defined by a mapping $\Sigma \to P(\Gamma^\star)$, where $\Sigma = \{a_1, \ldots, a_n\}$ and Γ are two finite alphabets, $f : a_i \to L_i$ where $\forall i \in [1; n]$, L_i is a finitary language over the alphabet Γ.
Now this mapping is extended in the usual manner to finite words:
$f(x(1) \ldots x(n)) = \{u_1 \ldots u_n \quad / \quad u_i \in f(x(i)), \forall i \in [1; n]\}$, where $x(1)$, ...,

$x(n)$ are letters in Σ. And to finitary languages $L \subseteq \Sigma^\star$: $f(L) = \cup_{x \in L} f(x)$. Let $\mathbb{C}$ be a family of languages, if $\forall i \in [1; n]$ the language L_i belongs to $\mathbb{C}$ the substitution f is called a $\mathbb{C}$-substitution.
Define then the operation $\square$ on families of languages: Let $\mathbb{C}$ and $\mathbb{D}$ be two families of (finitary) languages, then:

$$\mathbb{C} \,\square\, \mathbb{D} = \{f(L) \ / \ L \in \mathbb{C} \ \textit{and} \ f \ \textit{is a} \ \mathbb{D} - \textit{substitution}\}$$

Proposition 3.3

$$OCL = REG \square Rocl$$

In fact the operation of substitution gives rise to an infinite hierarchy of context free finitary languages defined as follows:

Definition 3.4 *Let $OCL(0) = REG$, $OCL(1) = OCL$ and $OCL(k+1) = OCL(k) \square OCL$ for $k \geq 1$.*

It is well known that the hierarchy given by the families of languages $OCL(k)$ is a strictly increasing hierarchy. And there exists a characterization of these languages by means of automata:

Proposition 3.5 ([ABB96]) *A language A is in $OCL(k)$ iff it is recognized (by accepting states) by a pushdown automaton such that, during any computation, the words in the pushdown store remain in a bounded language in the form $(z_k)^\star \ldots (z_2)^\star (z_1)^\star Z_0$, where $\{Z_0, z_1, \ldots, z_k\}$ is the pushdown alphabet.*
The union

$$ICL = \cup_{k \geq 1} OCL(k)$$

is called the family of iterated counter languages.

In order to generalize these results to languages of ω-words, we first define k-iterated counter pushdown machines:

Definition 3.6 *Let $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDM. If, during any computation, the words in the pushdown store remain in a bounded language in the form $(z_k)^\star \ldots (z_2)^\star (z_1)^\star Z_0$, where $\{Z_0, z_1, \ldots, z_k\}$ is the pushdown alphabet, the PDM M is said to be a k-iterated counter pushdown machine, and this leads in a natural manner to the definition of k-iterated counter pushdown automata (reading finite words) and k-iterated counter BPDA and k-iterated counter MPDA (reading infinite words).*

We have seen that the finitary languages accepted by k-iterated counter pushdown automata have a nice characterization: they form the class $OCL(k)$. Considering now automata reading ω-words, the following result holds:

Theorem 3.7 *Let $OCL(k)$ be the class of k-iterated counter (finitary) languages, for an integer $k \geq 1$. Then for any ω-language L the following three conditions are equivalent:*

1. *$L \in \omega - KC(OCL(k))$.*
2. *There exists a k-iterated counter $BPDA$ that accepts L.*
3. *There exists a k-iterated counter $MPDA$ that accepts L.*

Remark 3.8 *This result remains true for $k = 0$, and it is in fact the characterization Theorem 2.3 of ω-regular languages, with the convention that a 0-iterated counter PDA is a finite automaton (because the word in the pushdown store is then always Z_0 where Z_0 is the bottom symbol)*

Proof of $2 \leftrightarrow 3$**.** The k-iterated counter Büchi and Muller PDA considered here are non deterministic and then the expresssive power of k-iterated counter Büchi PDA is the same as the expresssive power of k-iterated counter Muller PDA. The idea of the proof is the same as for the general case of pushdown automata, [Sta97]. And in fact this is also true for a general storage type as considered in [EH93] and k-iterated counter storage type is a particular case of this result. □

Proof of $2 \leftrightarrow 1$. It is similar to the proof of the equivalence $2 \leftrightarrow 1$ of Theorem 2.9 given in [Sta97], replacing finitary context free languages by languages in $OCL(k)$ and pushdown automata by k-iterated counter PDA. □

Then we can let the following definition:

Definition 3.9 *An ω-language is a k-iterated counter ω-language (ω-k-ICL) iff it satisfies one of the conditions of the above Theorem. We denote $k - ICL_\omega$ the family of k-iterated counter ω-languages. An ω-language L is an iterated counter ω-language iff there exists an integer k such that $L \in k - ICL_\omega$. And*

$$ICL_\omega = \bigcup_{k \geq 1} k - ICL_\omega$$

is the family of iterated counter ω-languages.

Remark 3.10 *The class $k-ICL_\omega$ is defined by means of acceptation by* **non deterministic** *k-iterated counter PDA and thus it is closed under finite union. This property follows also from the characterization as the omega Kleene closure of the class $OCL(k)$. And then the whole class ICL_ω is also closed under finite union because the hierarchy of the classes $k - ICL_\omega$ is increasing as the hierarchy of the $OCL(k)$.*

It is proved in [CG77] that if $V \subseteq \Sigma^\star$ is a finitary language over the alphabet Σ and a is a new letter not in Σ, then the ω-language $V.a^\omega$ is an ω-CFL iff the language V is a context free (finitary) language.
This result can be extended to the class $k - ICL_\omega$ in the following form:

Proposition 3.11 *Let $V \subseteq \Sigma^\star$ be a finitary language over the alphabet Σ and a be a new letter not in Σ, then the ω-language $V.a^\omega$ is in $k - ICL_\omega$ iff the language V is in $OCL(k)$.*

Proof. In one direction it is obvious that if V is in $OCL(k)$, then the ω-language $V.a^\omega$ is in $\omega - KC(OCL(k)) = k - ICL_\omega$ because the language $\{a\}$ is in $OCL(k)$.
In the other direction, let us assume that $V.a^\omega$ is in $k - ICL_\omega$ where $V \subseteq \Sigma^\star$ and $a \notin \Sigma$. Then by Theorem 3.7 there exist some languages U_i and V_i in $OCL(k)$, $1 \leq i \leq n$, such that

$$V.a^\omega = \bigcup_{i=1}^{n} U_i.V_i^\omega$$

But then for each $i \in [1, n]$, $V_i \subseteq \{a^n \;/\; n \geq 1\}$.
Thus if U'_i is the image of U_i by the erasing morphism which just erases the letters a, it holds that

$$(\bigcup_{i=1}^{n} U'_i) = V$$

But $OCL(k)$ is closed under morphism because of the more general result that it is closed under regular substitution, then U'_i is in $OCL(k)$ for each integer $i \in [1, n]$. And $OCL(k)$ is closed under finite union (because it is defined by non deterministic machines) then $\bigcup_{i=1}^{n} U'_i = V$ is in $OCL(k)$. □

From this result one can deduce that the hierarchy of the classes $k - ICL_\omega$ is strictly increasing and strictly included in the class CFL_ω:

Theorem 3.12 *For each integer $k \geq 0$, the following inclusion is strict: $k - ICL_\omega \subsetneq (k+1) - ICL_\omega$, and the whole family of iterated counter ω-languages is strictly included into the family of omega context free languages: $ICL_\omega \subsetneq CFL_\omega$.*

Proof. It follows directly from the above proposition and the fact that for each integer $k \geq 0$, $OCL(k) \subsetneq OCL(k+1)$ and $ICL \subsetneq CFL$. $\square$

As in the general case of pushdown machines we can consider together finite and infinite runs of a k-iterated counter pushdown machine given with two state sets K_1 and K_2 (one is used for the acceptation of finite words by final states and the other for acceptation of ω-words by final states using a Büchi acceptance condition) and in a similar manner we let the following:

Definition 3.13 *Let X be a finite alphabet.*
A subset L of $X^{\leq\omega}$ is said to be an infinitary k-iterated counter language (or k-iterated counter $(\leq\omega)$-language) iff there exist a finitary language $L_1 \in OCL(k)$ and an ω-language $L_2 \in k-ICL_\omega$ such that $L = L_1 \cup L_2$.
The set of k-iterated counter $(\leq\omega)$-languages is denoted $k-ICL_{\leq\omega}$.

4 Borel and Wadge hierarchies

We assume the reader to be familiar with basic notions of topology which may be found in [LT94] [Kur66] and with the elementary theory of ordinals, including the operations of multiplication and exponentiation, which may be found in [Sie65].

Topology is an important tool for the study of ω-languages, and leads to characterization of several classes of ω-languages.
For a finite alphabet X, we consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The class of open sets of X^ω will be denoted by $\mathbf{G}$ or by $\mathbf{\Sigma^0_1}$. The class of closed sets will be denoted by $\mathbf{F}$ or by $\mathbf{\Pi^0_1}$.
Define now the next classes of the Borel Hierarchy:

Definition 4.1 *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ is the class of open sets of X^ω.
$\mathbf{\Pi^0_1}$ is the class of closed sets of X^ω.
$\mathbf{\Pi^0_2}$ or $\mathbf{G}_\delta$ is the class of countable intersections of open sets of X^ω.
$\mathbf{\Sigma^0_2}$ or $\mathbf{F}_\sigma$ is the class of countable unions of closed sets of X^ω.
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.
$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.
The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma^0_\alpha}$ and

$\mathbf{\Pi^0_\alpha}$, *for a countable ordinal* α, *are defined in the following way* [1]*:*
For a successor ordinal $(\alpha+1)$, *the definition is as above for* $(n+1)$. *And for a limit ordinal* α, $\mathbf{\Sigma^0_\alpha} = \mathbf{\Pi^0_\alpha} = \cup_{\gamma<\alpha}\mathbf{\Sigma^0_\gamma}$.

Recall some basic results about these classes:

Proposition 4.2 ([Mos80]) *a)* $\mathbf{\Sigma^0_\alpha} \cup \mathbf{\Pi^0_\alpha} \subsetneq \mathbf{\Sigma^0_{\alpha+1}} \cap \mathbf{\Pi^0_{\alpha+1}}$, *for each countable ordinal* α.

b) A set $W \subseteq X^\omega$ *is in the class* $\mathbf{\Sigma^0_\alpha}$ *if and only if its complement* W^- *is in the class* $\mathbf{\Pi^0_\alpha}$.

c) $\mathbf{\Sigma^0_\alpha} - \mathbf{\Pi^0_\alpha} \neq \emptyset$ *and* $\mathbf{\Pi^0_\alpha} - \mathbf{\Sigma^0_\alpha} \neq \emptyset$ *hold for every countable successor ordinal* α.

We shall say that a subset of X^ω is a Borel set of rank 1 iff it is in $\mathbf{\Sigma^0_1} \cup \mathbf{\Pi^0_1}$ and that it is a Borel set of rank $\alpha+1 \geq 2$, for a countable ordinal α, iff it is in $\mathbf{\Sigma^0_{\alpha+1}} \cup \mathbf{\Pi^0_{\alpha+1}}$ but not in $\mathbf{\Sigma^0_\alpha} \cup \mathbf{\Pi^0_\alpha}$.

Introduce now the Wadge Hierarchy which is in fact a huge refinement of the Borel hierarchy:

Definition 4.3 *For* $E \subseteq X^\omega$ *and* $F \subseteq Y^\omega$, E *is said Wadge reducible to* F $(E \leq_W F)$ *iff there exists a continuous function* $f : X^\omega \rightarrow Y^\omega$, *such that* $E = f^{-1}(F)$.
E *and* F *are Wadge equivalent iff* $E \leq_W F$ *and* $F \leq_W E$. *This will be denoted by* $E \equiv_W F$. *And we shall say that* $E <_W F$ *iff* $E \leq_W F$ *but not* $F \leq_W E$.
A set $E \subseteq X^\omega$ *is said to be self dual iff* $E \equiv_W E^-$, *and otherwise it is said to be non self dual.*

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The equivalence classes of $\equiv_W$ are called wadge degrees.
WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.

Remark 4.4 *In the above definition, we consider that a subset* $E \subseteq X^\omega$ *is given together with the alphabet* X.

Then we can define the Wadge class of a set F:

[1] In another presentation of the Borel hierarchy, as in [Mos80], when α is a limit ordinal, $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$) is the class we call here $\mathbf{\Sigma^0_{\alpha+1}}$ (respectively $\mathbf{\Pi^0_{\alpha+1}}$), and our class $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$), which is simply the union of the preceding ones, does not appear.

Definition 4.5 *Let F be a subset of X^ω. The wadge class of F is $[F]$ defined by: $[F] = \{E/E \subseteq Y^\omega$ for a finite alphabet Y and $E \leq_W F\}$.*

Recall that each Borel class $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ is a Wadge class.
And that a set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete set iff for any set $E \subseteq Y^\omega$, E is in $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$) iff $E \leq_W F$.
$\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete sets are thoroughly characterized in [Sta86].

Theorem 4.6 (Wadge) *Up to the complement and $\equiv_W$, the class of Borel subsets of X^ω, for X a finite alphabet, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d^0_W from WH onto $|WH| - \{0\}$, such that for all $A, B \in WH$:*
$d^0_W A < d^0_W B \leftrightarrow A <_W B$ and
$d^0_W A = d^0_W B \leftrightarrow [A \equiv_W B$ or $A \equiv_W B^-]$.

We shall here restrict our study to Borel sets of finite rank. And the Wadge hierarchy has then length ε^1 where ε^1 is the limit of the ordinals α_n defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer, ω_1 being the first non countable ordinal.

There is an effective version of the Wadge Hierarchy restricted to ω-regular languages:

Theorem 4.7 *For A and B some ω-regular sets, one can effectively decide whether $A \leq_W B$ and one can compute $d^0_W(A)$.*

The hierarchy obtained on ω-regular languages is now called the Wagner hierarchy and has length ω^ω. Wagner [Wag79] gave an automata structure characterization, based on notion of chain and superchain, for an automaton to be in a given class and then he got an algorithm to compute the Wadge degree of an ω-regular language. Wilke and Yoo proved in [WY95] that one can compute in polynomial time the Wadge degree of an ω-regular language. This hierarchy has been recently studied in [CP97] , [CP98] and [Sel98]. And it has an extension to omega deterministic context free languages which has length ω^{ω^2} [DFR99] [Dup99] [Fin99b].

The Wadge hierarchy restricted to ω-CFL is not effective: We have shown in [Fin99a] the following:

Theorem 4.8 *Let n be an integer ≥ 1. Then it is undecidable whether an effectively given ω-CFL is in the class $\mathbf{\Sigma^0_n}$ (repectively $\mathbf{\Pi^0_n}$).*

This result can be strengthened by showing the following:

Theorem 4.9 *Let B be a Borel set of finite rank such that $d_W^0 B = \alpha < \varepsilon^1$. Then it is undecidable whether an effectively given ω-CFL L is in the Wadge class $[B]$ of B, and it is undecidable whether $d_W^0(L) \leq \alpha$.*

Proof. As above, the ordinals α_n are defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer.
If $d_W^0(B) = \alpha < \varepsilon^1$, there exists an integer $n_B \geq 1$ such that $\alpha < \alpha_{n_B}$.
Recall that a Borel set $L \subseteq X^\omega$ is in the class $\mathbf{\Pi^0_{n+1}}$ or in the class $\mathbf{\Sigma^0_{n+1}}$ iff $d_W^0(L) \leq \alpha_n$ [Dup99].
Return now to the proof of Theorem 7.2 of [Fin99a]. Let n be an integer ≥ 1. We had found a family of omega context free languages

$$(A_{X,Y}^{\sim.n})^d = ((L_{X,Y} \cup \Sigma^\star)^{\sim.n})^d$$

over the alphabet $\{a, b, c, \twoheadleftarrow_1, \twoheadleftarrow_2, \ldots, \twoheadleftarrow_n, d\}$ such that $(A_{X,Y}^{\sim.n})^d$ is either $\{a, b, c, \twoheadleftarrow_1, \twoheadleftarrow_2, \ldots, \twoheadleftarrow_n, d\}^\omega$ or an ω-language which is neither a $\mathbf{\Pi^0_{n+1}}$-subset nor a $\mathbf{\Sigma^0_{n+1}}$-subset of $\{a, b, c, \twoheadleftarrow_1, \twoheadleftarrow_2, \ldots, \twoheadleftarrow_n, d\}^\omega$.
In the first case $d_W^0((A_{X,Y}^{\sim.n})^d) = d_W^0(\{a, b, c, \twoheadleftarrow_1, \twoheadleftarrow_2, \ldots, \twoheadleftarrow_n, d\}^\omega) = 1$, (because the Wadge degree of Σ^ω considered as an ω-language over the alphabet Σ is always 1).

And in the second case $d_W^0((A_{X,Y}^{\sim.n})^d) > \alpha_n$.
Take now the integer n_B and consider the family of omega context free languages

$$(A_{X,Y}^{\sim.n_B})^d$$

Then there are two cases:

a) $d_W^0((A_{X,Y}^{\sim.n_B})^d) = 1$

b) $d_W^0((A_{X,Y}^{\sim.n_B})^d) > \alpha_{n_B} > \alpha$

But one cannot decide which case holds. □

5 Operations on conciliating sets

5.1 Conciliating sets

We sometimes consider here subsets of $X^\star \cup X^\omega = X^{\leq\omega}$, for an alphabet X, which are called conciliating sets in [Dup99] [Dup95a]. In order to give a "normal form" of Borel sets in the Wadge hierarchy, J. Duparc studied the Conciliating hierarchy which is a hierarchy over conciliating sets closely related to the Wadge hierarchy. The two hierarchies are connected via the following correspondance:

First define A^d for $A \subseteq X_A^{\leq\omega}$ and d a letter not in X_A:

$$A^d = \{x \in (X_A \cup \{d\})^\omega \ / \ x(/d) \in A\}$$

where $x(/d)$ is the sequence obtained from x when removing every occurrence of the letter d.

Then for $A \subseteq X_A^{\leq\omega}$, A^d is always a non self dual subset of $(X_A \cup \{d\})^\omega$ and the correspondance $A \rightarrow A^d$ induces an isomorphism between the conciliating hierarchy and the Wadge hierarchy of non self dual sets. Hence we shall first concentrate on non self dual sets as in [Dup99] and we shall use the following definition of the Wadge degrees which is a slight modification of the previous one:

Definition 5.1

- $d_w(\emptyset) = d_w(\emptyset^-) = 1$
- $d_w(A) = sup\{d_w(B) + 1 \ / \ B$ non self dual and $B <_W A\}$ *(for either A self dual or not, $A >_W \emptyset$).*

Recall the definition of the conciliating degree of a conciliating set:

Definition 5.2 *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set over the alphabet X_A such that A^d is a Borel set. The conciliating degree of A is:*

$$d_c(A) = d_w(A^d)$$

Prove now some properties of the correspondance $A \rightarrow A^d$ when iterated counter languages are considered:

Proposition 5.3 *a) if $A \subseteq \Sigma^\star$ is a (finitary) language in $OCL(k)$, then A^d is in $k-ICL_\omega$.*

b) if $A \subseteq \Sigma^\omega$ is in $k-ICL_\omega$, then A^d is in $k-ICL_\omega$.

c) If A is the union of a finitary language in $OCL(k)$, and of an ω-language in $k-ICL_\omega$, over the same alphabet Σ, then A^d is a k-iterated counter ω-language over the alphabet $\Sigma \cup \{d\}$.

Proof of a).
Let $A \subseteq \Sigma^\star$ be a language in $OCL(k)$. Substitute first the language $(d^\star).a$ for each letter $a \in \Sigma$. In such a way we obtain another language A' in $OCL(k)$ because $OCL(k)$ is closed under substitution by regular languages and the languages $(d^\star).a$ are regular. Indeed $A^d = A'.d^\omega$ hence A^d is in $k - ICL_\omega$ because $\omega - KC(OCL(k)) = k - ICL_\omega$ by Theorem 3.7. $\square$

Proof of b).
Let $A \subseteq \Sigma^\omega$ be an ω-language in $k - ICL_\omega$. The ω-language A^d is obtained from A by substituting the language $(d^\star).a$ for each letter $a \in \Sigma$ in the words of A. But the class $k - ICL_\omega$ is closed under λ-free regular substitution because $OCL(k)$ is closed under regular substitution hence A^d is in $k - ICL_\omega$. $\square$

Proof of c).
Let A and B be subsets of $\Sigma^{\leq\omega}$ for a finite alphabet Σ. Then we easily see that if $C = A \cup B$, $C^d = A^d \cup B^d$ holds. c) is now an easy consequence of a) and b) because $k - ICL_\omega$ is closed under union. $\square$

And we now introduce several operations over conciliating sets:

5.2 Operation of sum

Definition 5.4 ([Dup99]) *Assume that $X_A \subseteq X_B$ and that $X_B - X_A$ contains at least two elements and that $\{X_+, X_-\}$ is a partition of $X_B - X_A$ in two non empty sets. Let $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$, then*

$$B+A = A \cup \{u.a.\beta \ / \ u \in X_A^\star, \ (a \in X_+ \text{ and } \beta \in B) \text{ or } (a \in X_- \text{ and } \beta \in B^-)\}$$

This operation is closely related to the ordinal sum as it is stated in the following:

Proposition 5.5 *Let $X_A \subseteq X_B$ and $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$ such that A^d and B^d are Borel sets. Then $(B + A)^d$ is a Borel set and:*

$$d_c(B + A) = d_c(B) + d_c(A)$$

Remark 5.6 *As indicated in Remark 5 of [Dup99], when $A \subseteq X_A^{\leq\omega}$ and X is a finite alphabet, it is easy to build $A' \subseteq (X_A \cup X)^{\leq\omega}$, such that $(A')^d \equiv_W A^d$. In fact A' can be defined as follows: for $\alpha \in (X_A \cup X)^{\leq\omega}$, let $\alpha \in A' \leftrightarrow \alpha' \in A$, where α' is α except each letter not in X_A is removed. Then in the sequel we assume that each alphabet is as enriched as desired, and in particular we can always define $B + A$ (or in fact another set C such that $C^d \equiv_W (B + A)^d$).*

Consider now conciliating sets which are union of a finitary language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$:

Proposition 5.7 *Let $X_A \subseteq X_B$ such that $X_B - X_A$ contains at least two elements and that $\{X_+, X_-\}$ is a partition of $X_B - X_A$ in two non empty sets. Assume $A \subseteq X_A^{\leq\omega}$ and $A, A^- \in k - ICL_{\leq\omega}$, and $B \subseteq X_B^{\leq\omega}$ and $B, B^- \in k - ICL_{\leq\omega}$, for an integer $k \geq 0$.*
Then $B + A$ is also in the form $D_1 \cup D_2$ where D_1 is in $OCL(k)$ and D_2 is in $k - ICL_\omega$ and its complement $X_B^{\leq\omega} - (B + A) = (X_B^\star - D_1) \cup (X_B^\omega - D_2)$ is also in that form.

Proof. Let A and B be two conciliating sets as in the hypothesis of the above proposition: assume $A = A_1 \cup A_2$ where A_1 and $X_A^\star - A_1$ are in $OCL(k)$ and A_2 and $X_A^\omega - A_2$ are in $k - ICL_\omega$, for an integer $k \geq 0$. And assume also that $B = B_1 \cup B_2$ where B_1 and $X_B^\star - B_1$ are in $OCL(k)$ and B_2 and $X_B^\omega - B_2$ are in $k - ICL_\omega$. By definition it holds that $B + A = A \cup X_A^\star.X_+.B \cup X_A^\star.X_-.B^-$.

Then the finite words in $B + A$ form the language $D_1 = A_1 \cup X_A^\star.X_+.B_1 \cup X_A^\star.X_-.B_1^-$ and the ω-words in $B + A$ form the ω-language $D_2 = A_2 \cup X_A^\star.X_+.B_2 \cup X_A^\star.X_-.B_2^-$.
$OCL(k)$ is closed under concatenation product and finite union hence $A_1 \cup X_A^\star.X_+.B_1 \cup X_A^\star.X_-.B_1^-$ is in $OCL(k)$. Similarly $k - ICL_\omega$ is closed under left concatenation by regular (finitary) languages and finite union hence $A_2 \cup X_A^\star.X_+.B_2 \cup X_A^\star.X_-.B_2^-$ is in $k - ICL_\omega$.

It remains to check that $X_B^{\leq\omega} - (B + A)$ is in the same form. But $X_B^{\leq\omega} - (B + A) = (X_B^\star - D_1) \cup (X_B^\omega - D_2)$, and $X_B^\star - D_1 = (X_A^\star - A_1) \cup X_A^\star.X_+.B_1^- \cup X_A^\star.X_-.B_1$ is in $OCL(k)$ because $(X_A^\star - A_1)$, B_1 and $X_B^\star - B_1$ are in $OCL(k)$. And $X_B^\omega - D_2 = (X_A^\omega - A_2) \cup X_A^\star.X_+.B_2^- \cup X_A^\star.X_-.B_2$, is in $k - ICL_\omega$ because $(X_A^\omega - A_2)$, B_2 and $X_B^\omega - B_2$ are in $k - ICL_\omega$.

5.3 Operation $A \rightarrow A^+$

Definition 5.8 *Let $A \subseteq X_A^{\leq\omega}$ and O_-, O_+ be two new letters not in X_A. Let $X = X_A \cup \{O_-, O_+\}$. Then A^+ is the conciliating set over the alphabet X defined by $A^+ = A \cup X^\star.O_+.A \cup X^\star.O_-.(X_A^{\leq\omega} - A)$.*

This operation is connected with the ordinal multiplication by ω_1:

Proposition 5.9 *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set over the alphabet X_A such that A^d is a Borel set. Then $(A^+)^d$ is a Borel set and:*

$$d_c(A^+) = d_c(A).\omega_1$$

Consider now conciliating sets which are unions of a finitary language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$:

Proposition 5.10 *Assume* $A = A_1 \cup A_2$ *where* A_1 *and* $X_A^\star - A_1$ *are in* $OCL(k)$ *and* A_2 *and* $X_A^\omega - A_2$ *are in* $k - ICL_\omega$*, for an integer* $k \geq 0$*. Then* A^+ *and* $(A^+)^-$ *are also unions of a finitary language in* $OCL(k)$ *and of an* ω*-language in* $k - ICL_\omega$*.*

Proof. Let $A = A_1 \cup A_2$ where A_1 and $X_A^\star - A_1$ are in $OCL(k)$ and A_2 and $X_A^\omega - A_2$ are in $k - ICL_\omega$. By definition $A^+ = A \cup X^\star.O_+.A \cup X^\star.O_-.(X_A^{\leq\omega} - A)$, so

$$A^+ = [A_1 \cup X^\star.O_+.A_1 \cup X^\star.O_-.(X_A^\star - A_1)] \cup [A_2 \cup X^\star.O_+.A_2 \cup X^\star.O_-.(X_A^\omega - A_2)]$$

But $OCL(k)$ is closed under finite union and concatenation and $k - ICL_\omega$ is closed under finite union and left concatenation by finitary languages in REG. This implies that A^+ is the union of a finitary language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$.

Consider now

$$(A^+)^- = (A^-)^+ \cup (X_A^\star.\{O_+, O_-\})^\omega$$

□

Where $(X_A^\star.\{O_+, O_-\})^\omega$ is the set of ω-words over the alphabet $X_A \cup \{O_+, O_-\}$ which contain infinitely many letters O_+ or O_-. This ω-language is an ω-regular language then it is in $k - ICL_\omega$ for any integer $k \geq 0$.
Then the same argument as for the case of A^+ shows that $(A^-)^+$ is in the form $V_1 \cup V_2$ with $V_1 \in OCL(k)$ and $V_2 \in k - ICL_\omega$ and then $(A^+)^-$ is in the same form because $k - ICL_\omega$ is closed under union. □

The two above operations $A, B \to B + A$ and $A \to A^+$ permit to obtain ω-languages C of Wadge degrees in the form

$$d_w(C) = \omega_1^{n_k}.m_k + \omega_1^{n_{k-1}}.m_{k-1} + \ldots + \omega_1^{n_1}.m_1$$

where $k > 0$ is an integer, $n_k > n_{k-1} > \ldots > n_1 \geq 0$ are integers and $m_k, m_{k-1}, \ldots, m_1$ are integers > 0.

For that it suffices to start with the emptyset $\emptyset$ (considered as a subset of $X^{\leq\omega}$ where X is an alphabet containing n letters, $n \geq 2$) and its complement $X^{\leq\omega}$. In fact the emptyset is given with the alphabet X so we start with infinitely many conciliating sets but for an alphabet X it always holds that:

$$d_c(\emptyset) = d_w((\emptyset)^d) = d_w((X^{\leq\omega})^d) = 1$$

Then take the closure of these conciliating sets under the two operations $A, B \rightarrow B + A$ and $A \rightarrow A^+$ and complementation. We obtain a family $\mathcal{C}_{\mathbf{0}}$ of conciliating sets closed under complementation such that, for $A \in \mathcal{C}_{\mathbf{0}}$, A^d is an ω-regular language and $d_c(A) = d_w(A^d)$ is in the above form.
It is well known that these degrees are exactly those of ω-regular languages. Thus in such a way, for each non self dual ω-regular language B, we obtain an ω-language A^d (with $A \in \mathcal{C}_{\mathbf{0}}$) such that A^d is Wadge equivalent to B.

The Wadge hierarchy of ω-regular languages has length ω^ω and it has also the same length when it is restricted to non self dual sets. Hence the family $\mathcal{C}_{\mathbf{0}}$ of conciliating sets provides a class ${\mathcal{C}_{\mathbf{0}}}^d = \{A^d \ / \ A \in \mathcal{C}_{\mathbf{0}}\}$ of ω-languages such that the length of the Wadge hierarchy of ${\mathcal{C}_{\mathbf{0}}}^d$ has length ω^ω.

5.4 Operation of multiplication by an ordinal $< \omega^\omega$

J. Duparc defined in [Dup99] another operation which is the multiplication by a countable ordinal, i.e. an ordinal $< \omega_1$. We shall restrict here the study to the operation of multiplication by an ordinal $< \omega^\omega$. And these operations may be defined by defining first the multiplication by the ordinal ω.

Definition 5.11 *Let $A \subseteq X_A^{\leq \omega}$ be a conciliating set over the alphabet X_A and O_+, O_- be two new letters not in X_A, then $A.\omega$ is defined over the alphabet $X_A \cup \{O_+, O_-\}$ by:*

$$A.\omega = \bigcup_{n \geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{\leq (n-1)}.X_A^\star.(O_+.A \cup O_-.A^-)$$

Thus in a (finite or infinite) word of $A.\omega$, the word has an initial prefix in the form $(O^+)^n.a$ for an integer $n \geq 1$ and a letter $a \in X_A$, and then there are at most n more letters from $\{O_+, O_-\}$ in the word and the last such letter determines whether the suffix following this last letter O_+ or O_- is in A or in A^-.

Prove now that k-iterated counter languages are closed under this operation:

In the following proposition, we consider first:

In a) $A \subseteq X_A^\star$, then $A.\omega$ is defined as in the preceding definition but with $A^- = X_A^\star - A$. Thus here $A.\omega$ is a set of **finite** words.

In b) $A \subseteq X_A^\omega$, then $A.\omega$ is defined as in the preceding definition but with $A^- = X_A^\omega - A$. Thus here $A.\omega$ is a set of **infinite** words.

Proposition 5.12 *a) If $A \subseteq X_A^\star$ is a (finitary) language in $OCL(k)$ such that $X_A^\star - A$ is also in $OCL(k)$, for an integer $k \geq 1$, then $A.\omega$ and $(A.\omega)^-$ are in $OCL(k)$.*

b) If $A \subseteq X_A^\omega$ is in $k - ICL_\omega$ and $X_A^\omega - A$ is in $k - ICL_\omega$, for an integer $k \geq 1$, then $A.\omega$ and $(A.\omega)^-$ are in $k - ICL_\omega$.

c) If $A \subseteq X_A^{\leq\omega}$ and $X_A^{\leq\omega} - A$ are unions of a language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$, then $A.\omega$ and $(A.\omega)^-$ are also in that form.

Proof of a).
Assume that $A \subseteq X_A^\star$ such that $A \in OCL(k)$ and $X_A^\star - A \in OCL(k)$. It is clear that the language

$$\bigcup_{n\geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{\leq(n-1)}.X_A^\star$$

is a one counter language in $OCL(1)$.(The counter is first increased of 1 when the one counter automaton reads a letter O_+ and after the first letter of X_A is read the counter is decreased when a letter O_+ or O_- is read). But $OCL(k)$ is closed under concatenation product and union hence $A.\omega$ is in $OCL(k)$.

From the definition of $A.\omega$, it holds that:

$$(A.\omega)^- = (O_+)^\star \cup ((O_+)^\star.O_- \cup X_A).(X_A \cup \{O_+, O_-\})^\star$$

$$\cup \bigcup_{n\geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{\geq(n+1)}.X_A^\star \quad \cup \quad (A^-).\omega$$

But $(O_+)^\star \cup ((O_+)^\star.O_- \cup X_A).(X_A \cup \{O_+, O_-\})^\star$ is a regular language and

$$\bigcup_{n\geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{\geq(n+1)}.X_A^\star$$

is a one counter language thus it is in $OCL(k)$ and so is $(A^-).\omega$ by similar arguments as for $A.\omega$, hence the language $(A.\omega)^-$ is in $OCL(k)$ because $OCL(k)$ is closed under union. □

Proof of b). Assume $A \subseteq X_A^\omega$ is in $k - ICL_\omega$ such that $X_A^\omega - A$ is also in $k - ICL_\omega$, for an integer $k \geq 1$. The proof that $A.\omega \in k - ICL_\omega$ is the same as for a) because $k - ICL_\omega$ is closed under left concatenation by languages in $OCL(k)$ (and also in $OCL(1)$ because $k \geq 1$ and $OCL(1) \subseteq OCL(k)$) and by finite union because

$$k - ICL_\omega = \omega - KC(OCL(k))$$

by Theorem 3.7.

From the definition of $A.\omega$, it holds that:

$$(A.\omega)^- = (O_+)^\omega \cup ((O_+)^\star.O_- \cup X_A).(X_A \cup \{O_+, O_-\})^\omega$$

$$\cup \bigcup_{n\geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{(n+1)}.(X_A \cup \{O_+, O_-\})^\omega \cup (A^-).\omega$$

But $(O_+)^\omega \cup ((O_+)^\star.O_- \cup X_A).(X_A \cup \{O_+, O_-\})^\omega$ is in REG_ω and

$$\bigcup_{n\geq 1} (O^+)^n.X_A.(X_A^\star.\{O_+, O_-\})^{(n+1)}.(X_A \cup \{O_+, O_-\})^\omega$$

and $(A^-).\omega$ are in $k - ICL_\omega$, hence $(A.\omega)^- \in k - ICL_\omega$ holds by finite union. □

Proof of c). Let $A = A_1 \cup A_2$ where A_1 and $X_A^\star - A_1$ are in $OCL(k)$ and A_2 and $X_A^\omega - A_2$ are in $k - ICL_\omega$, for an integer $k \geq 1$. Then, from the definition of $A.\omega$, it holds that:

$$A.\omega = A_1.\omega \cup A_2.\omega$$

$$(A.\omega)^- = [(X_A \cup \{O_+, O_-\})^\star - (A_1.\omega)] \cup [(X_A \cup \{O_+, O_-\})^\omega - (A_2.\omega)]$$

hence c) follows from a) and b). □

From this operation $A \to A.\omega$ over conciliating sets, we can inductively define the multiplication by an ordinal ω^n for an integer $n \geq 1$:

Definition 5.13 *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set over the alphabet X_A. Then $A.\omega^n$ is inductively defined by:*

a) $A.\omega$ is defined as above and

b) $A.\omega^{n+1} = (A.\omega^n).\omega$ for each integer $n \geq 1$.

In order to extend this definition to every non null ordinal $< \omega^\omega$, remark that it is well known that each non null ordinal $\alpha < \omega^\omega$ has a Cantor normal form [Sie65]:

$$\alpha = \omega^{n_k}.m_k + \omega^{n_{k-1}}.m_{k-1} + \ldots + \omega^{n_1}.m_1$$

where $k > 0$ is an integer, $n_k > n_{k-1} > \ldots > n_1 \geq 0$ are integers and $m_k, m_{k-1}, \ldots, m_1$ are integers > 0.

Definition 5.14 *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set over the alphabet X_A. Then $A.n$ is inductively defined by:*

a) $A.1 = A$

b) $A.(n+1) = (A.n) + A$ *for each integer* $n \geq 1$.

This allows to define $A.(\omega^{n_k}.m_k)$ for $n_k \geq 0$ and $m_k > 0$. And the operation of sum previously defined leads to the inductive definition of:

$$A.(\omega^{n_k}.m_k+\omega^{n_{k-1}}.m_{k-1}+\ldots+\omega^{n_1}.m_1) = A.(\omega^{n_k}.m_k)+A.(\omega^{n_{k-1}}.m_{k-1}+\ldots+\omega^{n_1}.m_1)$$

These operations $A \rightarrow A.\alpha$ satisfy the following:

Proposition 5.15 *Let α be a non null ordinal $< \omega^\omega$, then:*
If $A \subseteq X_A^{\leq\omega}$ and $X_A^{\leq\omega} - A$ are unions of a language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$ (i.e. are in $k - ICL_{\leq\omega}$) , then $A.\alpha$ and $(A.\alpha)^-$ are also in $k - ICL_{\leq\omega}$.

Proof. It follows from the similar properties for the operations of sum and of multiplication by ω of propositions 5.7 and 5.12, because of the inductive definition of the operations $A \rightarrow A.\alpha$, for $\alpha < \omega^\omega$, using the preceding operations $A, B \rightarrow A + B$ and $A \rightarrow A.\omega$. □

The operation $A \rightarrow A.\alpha$ is related with the ordinal multiplication by α:

Proposition 5.16 *Let α be a non null ordinal $< \omega^\omega$, and $A \subseteq X_A^{\leq\omega}$ such that A^d is a Borel set, then $(A.\alpha)^d$ is a Borel set and:*

$$d_c(A.\alpha) = d_c(A).\alpha$$

5.5 Operation of exponentiation

Definition 5.17 *Let X_A be a finite alphabet and $\twoheadleftarrow \notin X_A$, let $X = X_A \cup \{\twoheadleftarrow\}$. Let x be a finite or infinite word over the alphabet $X = X_A \cup \{\twoheadleftarrow\}$.*
Then $x^{\twoheadleftarrow}$ is inductively defined by:
$\lambda^{\twoheadleftarrow} = \lambda$,
For a finite word $u \in (X_A \cup \{\twoheadleftarrow\})^\star$:
$(u.a)^{\twoheadleftarrow} = u^{\twoheadleftarrow}.a$, *if* $a \in X_A$,
$(u. \twoheadleftarrow)^{\twoheadleftarrow} = u^{\twoheadleftarrow}$ *with its last letter removed if* $|u^{\twoheadleftarrow}| > 0$,
$(u. \twoheadleftarrow)^{\twoheadleftarrow} = \lambda$ *if* $|u^{\twoheadleftarrow}| = 0$,
and for u infinite:
$(u)^{\twoheadleftarrow} = \lim_{n\in\omega}(u[n])^{\twoheadleftarrow}$, *where, given β_n and u in $X_A^\star$,*
$u \sqsubseteq \lim_{n\in\omega} \beta_n \leftrightarrow \exists n \forall p \geq n \quad \beta_p[|u|] = u$.

Remark 5.18 *For $x \in X^{\leq\omega}$, $x^{\twoheadleftarrow}$ denotes the string x, once every $^{\twoheadleftarrow}$ occuring in x has been "evaluated" to the back space operation (the one familiar to your computer!), proceeding from left to right inside x. In other words $x^{\twoheadleftarrow} = x$ from which every interval of the form "$a \twoheadleftarrow$ " ($a \in X_A$) is removed.*

For example if $u = (a \twoheadleftarrow)^n$, for n an integer ≥ 1, or $u = (a \twoheadleftarrow)^\omega$, or $u = (a \twoheadleftarrow\twoheadleftarrow)^\omega$, then $(u)^{\twoheadleftarrow} = \lambda$,
if $u = (ab \twoheadleftarrow)^\omega$ then $(u)^{\twoheadleftarrow} = a^\omega$,
if $u = bb(\twoheadleftarrow a)^\omega$ then $(u)^{\twoheadleftarrow} = b$.

We can now define the operation $A \to A^\sim$ of exponentiation of conciliating sets:

Definition 5.19 *For $A \subseteq X_A^{\leq\omega}$ and $\twoheadleftarrow \notin X_A$, let $X = X_A \cup \{\twoheadleftarrow\}$ and*
$A^\sim = \{x \in (X_A \cup \{\twoheadleftarrow\})^{\leq\omega} / x^{\twoheadleftarrow} \in A\}$.

The operation $\sim$ is monotone with regard to the Wadge ordering and produce some sets of higher complexity, in the following sense:

Theorem 5.20 (Duparc [Dup99]) *a) For $A \subseteq X_A^{\leq\omega}$ and $B \subseteq X_B^{\leq\omega}$, A^d and B^d borel sets, $A^d \leq_W B^d \leftrightarrow (A^\sim)^d \leq_W (B^\sim)^d$.*

b) If $A^d \subseteq (X_A \cup \{d\})^\omega$ is a $\mathbf{\Sigma^0_n}$-complete (respectively $\mathbf{\Pi^0_n}$-complete) set (for an integer $n \geq 1$), then $(A^\sim)^d$ is a $\mathbf{\Sigma^0_{n+1}}$-complete (respectively $\mathbf{\Pi^0_{n+1}}$-complete) set.

Recall now the notion of cofinality of an ordinal which is an important notion in set theory [CK73]. Let α be a limit ordinal, the cofinality of α, denoted

$cof(\alpha)$, is the least ordinal β such that there exists a strictly increasing sequence of ordinals $(\alpha_i)_{i<\beta}$, of length β, such that

$$\forall i < \beta \qquad \alpha_i < \alpha \quad \text{and}$$

$$\sup_{i<\beta} \alpha_i = \alpha$$

This definition is usually extended to 0 and to the successor ordinals:

$$cof(0) = 0 \text{ and } cof(\alpha + 1) = 1 \text{ for every ordinal } \alpha$$

The cofinality of a limit ordinal is always a limit ordinal satisfying:

$$\omega \leq cof(\alpha) \leq \alpha$$

$cof(\alpha)$ is in fact a cardinal [CK73]. Then if the cofinality of a limit ordinal α is $\leq \omega_1$, only the following cases may happen:

$$cof(\alpha) = \omega \text{ or } cof(\alpha) = \omega_1$$

In this paper we shall not have to consider larger cofinalities.

We can now state that the operation of exponentiation of conciliating sets is closely related to ordinal exponentiation of base ω_1:

Theorem 5.21 (Duparc [Dup99]) *Let $A \subseteq X_A^{\leq\omega}$ be a conciliating set such that A^d is a Borel set and $d_c(A) = d_w(A^d) = \alpha + n$ with α a limit ordinal and n an integer ≥ 0. Then $(A^\sim)^d$ is a Borel set and there are three cases:*

- *a) If $\alpha = 0$, then $d_c(A^\sim) = (\omega_1)^{d_c(A)-1}$*
- *b) If α has cofinality ω, then $d_c(A^\sim) = (\omega_1)^{d_c(A)+1}$*
- *c) If α has cofinality ω_1, then $d_c(A^\sim) = (\omega_1)^{d_c(A)}$*

Consider now this operation $\sim$ with regard to k-iterated counter languages:

Theorem 5.22 *Whenever $A \subseteq X_A^\omega$ is in $k-ICL_\omega$, then $A^\sim \subseteq (X_A \cup \{\twoheadleftarrow\})^\omega$ is in $(k+1) - ICL_\omega$.*

Proof. An ω-word $\sigma \in A^{\sim}$ may be considered as an ω-word $\sigma^{\twoheadleftarrow} \in A$ to which we possibly add, before the first letter $\sigma^{\twoheadleftarrow}(1)$ of $\sigma^{\twoheadleftarrow}$ (respectively between two consecutive letters $\sigma^{\twoheadleftarrow}(n)$ and $\sigma^{\twoheadleftarrow}(n+1)$ of $\sigma^{\twoheadleftarrow}$), a finite word v_1 (respectively v_{n+1}) where:
v_{n+1} belongs to the context free (finitary) language L_3 generated by the context free grammar with the following production rules:
$S \rightarrow aS \twoheadleftarrow S$ with $a \in X_A$,
$S \rightarrow a \twoheadleftarrow S$ with $a \in X_A$,
$S \rightarrow \lambda$ (λ being the empty word).
this language L_3 corresponds to words where every letter of X_A has been removed after using the back space operation.
And v_1 belongs to the finitary language $L_4 = (\twoheadleftarrow)^{\star}.(L_3.(\twoheadleftarrow)^{\star})^{\star}$. This language corresponds to words where every letter of X_A has been removed after using the back space operation and this operation maybe has been used also when there was not any letter to erase. L_3 is a one counter language i.e. L_3 is in OCL (during a reading of a word the counter is increased when a letter of X_A is read and it is decreased when a letter $\twoheadleftarrow$ is read). And for $a \in X_A$, the language $L_3.a$ is also accepted by a one counter automaton. L_4 is also in OCL because the class OCL is closed under star operation and concatenation product.
Then we can state the following:

Lemma 5.23 *Whenever $A \subseteq X_A^{\omega}$, the ω-language $A^{\sim} \subseteq (X_A \cup \{\twoheadleftarrow\})^{\omega}$ is obtained by substituting in A the language $L_3.a$ for each letter $a \in X_A$, where L_3 is the one counter language defined above, and then making a left concatenation by the language L_4.*

Let now A be an ω-language in $k-ICL_{\omega}$, given by $A = \bigcup_{i=1}^{n} U_i.V_i^{\omega}$ where U_i and V_i are in $OCL(k)$. Then $A^{\sim} = \bigcup_{i=1}^{n}(L_4.U_i').V_i'^{\omega}$, where U_i' (respectively V_i') is obtained by substituting the language $L_3.a$ to each letter $a \in X_A$ in U_i (respectively V_i).
It holds that $OCL(k)\square OCL = OCL(k+1)$, so U_i' and V_i' are in $OCL(k+1)$, and so is the language $(L_4.U_i')$ by concatenation product (because $L_4 \in OCL \subseteq OCL(k+1)$, and $OCL(k+1)$ is closed under concatenation product). Hence the ω-language $A^{\sim}$ is in $(k+1)-ICL_{\omega}$, because $\omega - KC(OCL(k+1)) = (k+1) - ICL_{\omega}$.

Consider now subsets of $X^{\leq\omega}$ in the form $A \cup B$, where A is a finitary language in $OCL(k)$ and B is an ω-language in $k - ICL_{\omega}$. Remark that A and B should not be accepted by the same pushdown automaton (but it may be). Prove then the following.

Proposition 5.24 *If $C = A \cup B$, where A is a language in $OCL(k)$ and B is an ω-language in $k - ICL_\omega$ over the same alphabet $X_A = X_B$, then $C^\sim$ is the union of a finitary language in $OCL(k+1)$ and of an ω-language in $(k+1) - ICL_\omega$ over the alphabet $X_A \cup \{\twoheadleftarrow\}$.*

Proof. It is easy to see from the definition of the operation of exponentiation of sets that if $C = A \cup B$ then: $C^\sim = A^\sim \cup B^\sim$.
But if B is a k-iterated counter ω-language over $X_B = X_A$, then by Theorem 5.22 $B^\sim$ is a $k+1$-iterated counter ω-language D_1.

Consider now the set $A^\sim$: This subset of $(X_A \cup \{\twoheadleftarrow\})^{\leq\omega}$ is constituted of finite and infinite words. Let h be the substitution: $X \to P((X_A \cup \{\twoheadleftarrow\})^\star)$ defined by $a \to a.L_3$ where L_3 is the one counter language defined above. Then it is easy to see that the finite words are obtained by substituting in A the language $a.L_3$ for each letter $a \in X_A$ and concatenating on the left by the language L_4.
But after substitution we obtain a language in $OCL(k{+}1)$ because $OCL(k)\Box OCL = OCL(k+1)$, and then by concatenation by the language L_4 which is in OCL we obtain a language D_2 which is also in $OCL(k+1)$.

The infinite words in $A^\sim$ constitutes the ω-language
$D_2.(L_3 - \{\lambda\})^\omega$ if $\lambda \notin A$, and
$D_2.(L_3 - \{\lambda\})^\omega \cup (L_4 - \{\lambda\})^\omega$ if $\lambda \in A$,

The languages $L_4 - \{\lambda\}$ and $L_3 - \{\lambda\}$ are one counter languages, thus the set of infinite words in $A^\sim$ is a $(k+1)$-iterated counter ω-language D_3 because $\omega - KC(OCL(k+1)) \subseteq (k+1) - ICL_\omega$ by Theorem 3.7. Then:

$$A^\sim = D_1 \cup D_2 \cup D_3$$

But $(k+1) - ICL_\omega$ is closed under union hence $D_1 \cup D_3$ is in $(k+1) - ICL_\omega$.
□

Remark 5.25 *It is easy to see from the definition of $A^\sim$ that whenever $A \subseteq X_A^{\leq\omega}$ it holds that:*

$$(X_A \cup \{\twoheadleftarrow\})^{\leq\omega} - A^\sim = (X_A^{\leq\omega} - A)^\sim$$

hence if A and A^- are unions of a language in $OCL(k)$ and of an ω-language in $k - ICL_\omega$, $A^\sim$ and $(A^\sim)^-$ are unions of a language in $OCL(k+1)$ and of an ω-language in $(k+1) - ICL_\omega$, by proposition 5.24.

6 Conciliating hierarchy of infinitary context free languages

In this section we study the conciliating hierarchy of infinitary context free languages.

We denote $Co-k-ICL_\omega$ (respectively $Co-k-ICL_{\leq\omega}$) the class of complements of ω-languages (respectively $(\leq\omega)$-languages) which are in $k-ICL_\omega$ (respectively $k-ICL_{\leq\omega}$) and similarly we denote $Co-CFL_\omega$ (respectively $Co-CFL_{\leq\omega}$) the class of complements of omega (respectively infinitary) context free languages.

Then we can summarize the preceding results:

Proposition 6.1 *a) For each integer $k \geq 0$, the class $(k-ICL_{\leq\omega}) \cap (Co-k-ICL_{\leq\omega})$ is closed under the operations $A, B \to A+B$, $A \to A^+$.*

b) For each integer $k \geq 1$, the class $(k-ICL_{\leq\omega}) \cap (Co-k-ICL_{\leq\omega})$ is closed under the operations $A \to A.\alpha$, for $\alpha < \omega^\omega$.

c) For each integer $k \geq 0$, if $A \in (k-ICL_{\leq\omega}) \cap (Co-k-ICL_{\leq\omega})$ the $(\leq\omega)$-language $A^\sim$ is in $((k+1)-ICL_{\leq\omega}) \cap (Co-(k+1)-ICL_{\leq\omega})$.

Introduce now some notations for ordinals obtained by iterating the operation of exponentiation of base ω: i.e. the operation $\alpha \to \omega^\alpha$ for α ordinal. We denote $\omega(1) = \omega$ and for an integer $n \geq 1$, $\omega(n+1) = \omega^{\omega(n)}$:

$$\omega(n) = \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{n}$$

Then the limit of the ordinals $\omega(n)$ which is also the upper bound of the ordinals $\omega(n)$ is the well known Cantor ordinal ε_0. It is the first fixed point of the operation of exponentiation of base ω.

Now we can state the main result about the conciliating hierarchy of infinitary context free languages.

Theorem 6.2 *a) For each integer $k \geq 0$, the length of the conciliating hierarchy of $(\leq\omega)$-languages in $(k-ICL_{\leq\omega}) \cap (Co-k-ICL_{\leq\omega})$ is an ordinal greater than $\omega(k+2)$.*

b) the length of the conciliating hierarchy of iterated counter (and $Co-$iterated counter) infinitary languages is an ordinal greater than ε_0.

Corollary 6.3 *The length of the conciliating hierarchy of infinitary languages in $CFL_{\leq\omega} \cap Co - CFL_{\leq\omega}$ is greater than ε_0.*

Proof of a). We reason by induction on the integer k. The result has been already proved for the case $k = 0$.

In order to prove a) for $k > 0$, we shall use only the operation of sum and the operation of exponentiation $A \to A^\sim$.
Recall that if $A \subseteq X_A^{\leq\omega}$ is a conciliating set such that A^d is a Borel set and $d_c(A) = d_w(A^d) = \alpha + n$ with α a limit ordinal and n an integer ≥ 0, then there are three cases:

a) If $\alpha = 0$, then $d_c(A^\sim) = (\omega_1)^{d_c(A)-1}$

b) If α has cofinality ω, then $d_c(A^\sim) = (\omega_1)^{d_c(A)+1}$

c) If α has cofinality ω_1, then $d_c(A^\sim) = (\omega_1)^{d_c(A)}$

We have already obtained a family $\mathcal{C}_\mathbf{0}$ of conciliating sets in $(0 - ICL_{\leq\omega}) \cap (Co - 0 - ICL_{\leq\omega})$, closed under complementation, such that, for $A \in \mathcal{C}_\mathbf{0}$, $d_c(A)$ is in the following form:

$$d_c(A) = \omega_1^{n_j}.m_j + \omega_1^{n_{j-1}}.m_{j-1} + \ldots + \omega_1^{n_1}.m_1$$

where $j > 0$ is an integer, $n_j > n_{j-1} > \ldots > n_1 \geq 0$ are integers and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Then for $A \in \mathcal{C}_\mathbf{0}$, $d_c(A)$ is an ordinal in the form $\alpha + n$, with $\alpha = 0$ or α a limit ordinal of cofinality ω_1, and n an integer ≥ 0.

Then if $A \in \mathcal{C}_\mathbf{0}$, there are two cases:

a) If $d_c(A) = n$, n being an integer ≥ 1, then $d_c(A^\sim) = \omega_1^{n-1}$

b) If $d_c(A) = \alpha + n$ with α a limit ordinal of cofinality ω_1, then $d_c(A^\sim) = (\omega_1)^{d_c(A)}$

So we see that $d_c(A^\sim)$ may take the value 1 and all the values ω_1^β for $\beta \in \{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{0}\} = \mathcal{D}_\mathbf{0}$.

From the closure properties of proposition 6.1, we can infer that $(1 - ICL_{\leq\omega}) \cap (Co - 1 - ICL_{\leq\omega})$ contains all $(\leq \omega)$-languages in the form:

$$(A_j)^\sim.n_j + (A_{j-1})^\sim.n_{j-1} + \ldots + (A_1)^\sim.n_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}_\mathbf{0}$, and $n_1, n_2, \ldots, n_j$ are integers ≥ 1.

The length of the conciliating hierarchy of $\mathcal{C}_\mathbf{0}$ is ω^ω and there exists a strictly increasing isomorphism:

$$\phi_0 : \quad \{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{0}\} \longrightarrow \omega^\omega - \{0\}$$

$$\omega_1^{n_j}.m_j + \omega_1^{n_{j-1}}.m_{j-1} + \ldots + \omega_1^{n_1}.m_1 \longrightarrow \alpha = \omega^{n_j}.m_j + \omega^{n_{j-1}}.m_{j-1} + \ldots + \omega^{n_1}.m_1$$

where $j > 0$ is an integer, $n_j > n_{j-1} > \ldots > n_1 \geq 0$ are integers and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

This isomorphism is easily extended to a strictly increasing isomorphism:

$$\bar{\phi_0} : \quad \{0\} \cup \{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{0}\} \longrightarrow \omega^\omega$$

$$\alpha \neq 0 \longrightarrow \phi_0(\alpha)$$

$$0 \longrightarrow 0$$

Define $\mathcal{C}_\mathbf{1}$ as the family containing all conciliating sets in the following form and their complements:

$$(A_j)^\sim.n_j + (A_{j-1})^\sim.n_{j-1} + \ldots + (A_1)^\sim.n_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}_\mathbf{0}$, and $n_1, n_2, \ldots, n_j$ are integers ≥ 1.

Then $\mathcal{C}_\mathbf{1} \subseteq (1 - ICL_{\leq\omega}) \cap (Co - 1 - ICL_{\leq\omega})$
We shall prove that the length of the conciliating hierarchy of $\mathcal{C}_\mathbf{1}$ is greater than $\omega(3)$.

Remark first that for $A \in \mathcal{C}_\mathbf{1}$, $d_c(A)$ is in the following form:

$$d_c(A) = \omega_1^{\alpha_j}.m_j + \omega_1^{\alpha_{j-1}}.m_{j-1} + \ldots + \omega_1^{\alpha_1}.m_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{0}\} \cup \{0\}$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Consider now the Cantor normal form of a non null ordinal

$$\alpha < \omega^{\omega^\omega} = \omega(3)$$

Such an ordinal α can be written in the form:

$$\alpha = \omega^{\delta_j}.m_j + \omega^{\delta_{j-1}}.m_{j-1} + \ldots + \omega^{\delta_1}.m_1$$

where $j > 0$ is an integer, $\delta_j > \delta_{j-1} > \ldots > \delta_1$ are ordinals $< \omega^\omega$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Now it is easy to see that there exists a strictly increasing isomorphism:

$$\phi_1 : \quad \{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{1}\} \longrightarrow \omega^{\omega^\omega} - \{0\}$$

$$\omega_1^{\alpha_j}.m_j + \omega_1^{\alpha_{j-1}}.m_{j-1} + \ldots + \omega_1^{\alpha_1}.m_1 \longrightarrow \alpha = \omega^{\bar{\phi_0}(\alpha_j)}.m_j + \omega^{\bar{\phi_0}(\alpha_{j-1})}.m_{j-1} + \ldots + \omega^{\bar{\phi_0}(\alpha_1)}.m_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{0}\} \cup \{0\}$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Hence the length of the conciliating hierarchy of infinitary languages in $\mathcal{C}_\mathbf{1}$ and also in $(1 - ICL_{\leq\omega}) \cap (Co - 1 - ICL_{\leq\omega})$ is greater than ω^{ω^ω} (the order type of the set of ordinals $\omega^{\omega^\omega} - \{0\}$ is given by the ordinal ω^{ω^ω}).

Now we see that we can iterate this proof:

Assume that we have already obtained a family $\mathcal{C}_\mathbf{k}$ of conciliating sets in $(k - ICL_{\leq\omega}) \cap (Co - k - ICL_{\leq\omega})$, closed under complementation, such that, for $A \in \mathcal{C}_\mathbf{k}$, $d_c(A)$ is an ordinal in the form $\alpha + n$, with $\alpha = 0$ or α a limit ordinal of cofinality ω_1, and n an integer ≥ 0. And assume also that there exists a strictly increasing isomorphism:

$$\bar{\phi_k} : \quad \{0\} \cup \{d_c(A) \ / \ A \in \mathcal{C}_\mathbf{k}\} \longrightarrow \omega(k+2)$$

Define $\mathcal{C}_\mathbf{k+1}$ as the family containing all conciliating sets in the following form and their complements:

$$(A_j)^\sim.n_j + (A_{j-1})^\sim.n_{j-1} + \ldots + (A_1)^\sim.n_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}_\mathbf{k}$, and $n_1, n_2, \ldots, n_j$ are integers ≥ 1.

Then $\mathcal{C}_\mathbf{k+1} \subseteq ((k+1) - ICL_{\leq\omega}) \cap (Co - (k+1) - ICL_{\leq\omega})$
We shall prove that the length of the conciliating hierarchy of $\mathcal{C}_\mathbf{k+1}$ is greater than $\omega(k+3)$.

Remark first that for $A \in \mathcal{C}_\mathbf{k+1}$, $d_c(A)$ is in the following form:

$$d_c(A) = \omega_1^{\alpha_j}.m_j + \omega_1^{\alpha_{j-1}}.m_{j-1} + \ldots + \omega_1^{\alpha_1}.m_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{k}}\} \cup \{0\}$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Consider now the Cantor normal form of a non null ordinal

$$\alpha < \omega(k+3)$$

Such an ordinal α can be written in the form:

$$\alpha = \omega^{\delta_j}.m_j + \omega^{\delta_{j-1}}.m_{j-1} + \ldots + \omega^{\delta_1}.m_1$$

where $j > 0$ is an integer, $\delta_j > \delta_{j-1} > \ldots > \delta_1$ are ordinals $< \omega(k+2)$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Now it is easy to see that there exists a strictly increasing isomorphism:

$$\phi_{k+1} : \quad \{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{k+1}}\} \longrightarrow \omega(k+3) - \{0\}$$

$$\omega_1^{\alpha_j}.m_j + \omega_1^{\alpha_{j-1}}.m_{j-1} + \ldots + \omega_1^{\alpha_1}.m_1 \longrightarrow \alpha = \omega^{\bar{\phi}_k(\alpha_j)}.m_j + \omega^{\bar{\phi}_k(\alpha_{j-1})}.m_{j-1} + \ldots + \omega^{\bar{\phi}_k(\alpha_1)}.m_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{k}}\} \cup \{0\}$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

Hence the length of the conciliating hierarchy of infinitary languages in $\mathcal{C}_{\mathbf{k+1}}$ and also in $((k+1) - ICL_{\leq\omega}) \cap (Co - (k+1) - ICL_{\leq\omega})$ is greater than $\omega(k+3)$ (the order type of the set of ordinals $\omega(k+3) - \{0\}$ is given by the ordinal $\omega(k+3)$).

And we can define the isomorphism $\bar{\phi}_{k+1}$ from ϕ_{k+1} then this ends the proof by induction on the integer k. □

7 Wadge hierarchy of omega context free languages

We consider now ω-languages. Recall that the operation $A \rightarrow A^d$ over conciliating sets has the following property:

If $A \in (k{-}ICL_{\leq\omega}) \cap (Co{-}k{-}ICL_{\leq\omega})$ then $A^d \in (k{-}ICL_{\omega}) \cap (Co{-}k{-}ICL_{\omega})$. And if A^d is a Borel set, it holds that: $d_w(A^d) = d_c(A)$.

Hence the following results can be derived from the corresponding study of the conciliating hierarchy of iterated counter ($\leq \omega$)-languages:

Theorem 7.1 *a) For each integer $k \geq 0$, the length of the Wadge hierarchy of ω-languages in $(k - ICL_\omega) \cap (Co - k - ICL_\omega)$ is an ordinal greater than $\omega(k+2)$.*

b) the length of the Wadge hierarchy of iterated counter (and $Co-$iterated counter) ω-languages is an ordinal greater than ε_0.

Corollary 7.2 *The length of the Wadge hierarchy of context free ω-languages is greater than ε_0.*

Proof. Recall that we had obtained in proof of Theorem 6.2, for each integer $k \geq 0$, a family $\mathcal{C}_\mathbf{k}$ of conciliating sets in $(k - ICL_{\leq\omega}) \cap (Co - k - ICL_{\leq\omega})$, closed under complementation, such that the conciliating hierarchy restricted to $\mathcal{C}_\mathbf{k}$ has length $\omega(k+2)$.

Let $\mathcal{C}_\mathbf{k}^\mathbf{d} = \{A^d \ / \ A \in \mathcal{C}_\mathbf{k}\}$. Then $\mathcal{C}_\mathbf{k}^\mathbf{d} \subseteq (k - ICL_\omega) \cap (Co - k - ICL_\omega)$.
And the relation $d_w(A^d) = d_c(A)$ implies that the Wadge hierarchy of ω-languages in $\mathcal{C}_\mathbf{k}^\mathbf{d}$ has length $\omega(k+2)$.

b) follows from a) and the definition of the ordinal ε_0. □

Remark 7.3 *In fact the ω-languages in $\mathcal{C}_\mathbf{k}^\mathbf{d}$ are non self dual hence the wadge hierarchy of non self dual sets in $CFL_\omega \cap Co-CFL_\omega$ has length $\geq \varepsilon_0$. And we can generate self dual omega context free languages from non self dual ones:*

Definition 7.4 *Let $A \subseteq X_A^\omega$ and let $\{X_+, X_-\}$ be a partition of X_A into two non empty sets. The ω-language $S(A)$ is defined by: $S(A) = X_+.A \cup X_-.A^-$.*

Proposition 7.5 ([Dup99]) *Let $A \subseteq X_A^\omega$ be a non self dual Borel set. Then $S(A)$ is a self dual Borel set and it is the $<_W$-least above A (and A^-).*

And, with regard to iterated counter languages, it holds that:

Proposition 7.6 *If $A \subseteq X_A^\omega$ is in $(k - ICL_\omega) \cap (Co - k - ICL_\omega)$, then $S(A)$ and $S(A)^-$ are in $(k - ICL_\omega) \cap (Co - k - ICL_\omega)$.*

Proof. Assume $A \subseteq X_A^\omega$ is in $(k - ICL_\omega) \cap Co - k - ICL_\omega)$. By definition $S(A) = X_+.A \cup X_-.A^-$ but $k - ICL_\omega$ is closed under left concatenation by regular languages and union hence $S(A)$ is in $k - ICL_\omega$ and $S(A)^- = X_+.A^- \cup X_-.A$ hence by a similar argument $S(A)^-$ is in $k - ICL_\omega$. $\square$

Then we can deduce the following:

Theorem 7.7 *a) For each integer $k \geq 0$, the length of the Wadge hierarchy of* **non self dual** *ω-languages in $(k - ICL_\omega) \cap (Co - k - ICL_\omega)$ is an ordinal greater than $\omega(k+2)$.*

b) For each integer $k \geq 0$, the length of the Wadge hierarchy of **self dual** *ω-languages in $(k - ICL_\omega) \cap (Co - k - ICL_\omega)$ is an ordinal greater than $\omega(k+2)$.*

Corollary 7.8 *a) The length of the Wadge hierarchy of* **non self dual** *context free ω-languages is greater than ε_0.*

b) The length of the Wadge hierarchy of **self dual** *context free ω-languages is greater than ε_0.*

Remark 7.9 *Up to now we have just used the operations $A, B \rightarrow A + B$, $A \rightarrow A^+$ and $A \rightarrow A^\sim$ to obtain our results on the length of the studied hierarchies. So natural questions now arise: what about the operation of multiplication by an ordinal $< \omega^\omega$? Could we improve the preceding results by considering this new operation?*

In fact we can obtain many more Wadge degrees in such a way. For example there exists in $1 - ICL_\omega$, i.e. in the class of one counter ω-languages, an ω-language of Wadge degree α, for each ordinal $\alpha < \omega^\omega$. But without this operation we can only obtain some ω-languages of Wadge degree $< \omega$ or of Wadge degree $\geq \omega_1$. Recall that a Borel set has Wadge degree $\geq \omega_1$ if and only if it is not in $\mathbf{\Sigma^0_2} \cap \mathbf{\Pi^0_2}$, [Dup99]. Then we deduce the following:

Proposition 7.10 *The length of the Wadge hierarchy of one counter ω-languages which are in $\mathbf{\Sigma^0_2} \cap \mathbf{\Pi^0_2}$ is greater than ω^ω.*

And in a similar manner we can obtain many more Wadge degrees for greater ordinals.

On the other hand, the Wadge hierarchy of **deterministic** context free ω-languages has been determined: it has length $\omega^{(\omega^2)}$, [DFR99] [Dup99][Fin99b].

And the Wadge degrees of deterministic context free ω-languages are in the following form:

$$d_w(A) = \omega_1^{n_j}.\delta_j + \omega_1^{n_{j-1}}.\delta_{j-1} + \ldots + \omega_1^{n_1}.\delta_1$$

where $j > 0$ is an integer, $n_j > n_{j-1} > \ldots > n_1$ are integers ≥ 0, and $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are non null ordinals $< \omega^\omega$.

Then we see that one can obtain non self dual one counter ω-languages of each such degree and the self dual ones are generated by the preceding operation $A \to S(A)$. Then the hierarchy of one counter ω-languages is strictly larger than the hierarchy of deterministic context free ω-languages: there exists some one counter ω-languages which are not in $\mathbf{\Sigma^0_3}$, for example an ω-language of Wadge degree $\omega_1^{\omega_1^2}$ (because a Borel set is in $\mathbf{\Sigma^0_3} \cup \mathbf{\Pi^0_3}$ iff its Wadge degree is $\leq \omega_1^{\omega_1}$, [Dup99]) but deterministic context free ω-languages are boolean combinations of $\mathbf{\Sigma^0_2}$-sets, hence in $\mathbf{\Sigma^0_3}$. And the lengths of the hierarchies are respectively $\omega^{(\omega^2)}$ and $\geq \omega^{\omega^\omega}$. So we can state the:

Proposition 7.11 *For each deterministic context free ω-language L, there exists a one counter ω-language L_1 which is Wadge equivalent to L. But the converse is not true.*

Consider now the lower bounds for the lengths of the hierarchies we have studied. Can we get better results?

Recall we have inductively defined in section 6 the class $\mathcal{C}_{\mathbf{k+1}} \subseteq ((k+1) - ICL_{\leq\omega}) \cap (Co - (k+1) - ICL_{\leq\omega})$ as the family containing all conciliating sets in the following form and their complements:

$$(A_j)^\sim.n_j + (A_{j-1})^\sim.n_{j-1} + \ldots + (A_1)^\sim.n_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}_{\mathbf{k}}$, and $n_1, n_2, \ldots, n_j$ are integers ≥ 1. Then for $A \in \mathcal{C}_{\mathbf{k+1}}$, $d_c(A)$ was in the following form:

$$d_c(A) = \omega_1^{\alpha_j}.m_j + \omega_1^{\alpha_{j-1}}.m_{j-1} + \ldots + \omega_1^{\alpha_1}.m_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{k}}\} \cup \{0\}$ and $m_j, m_{j-1}, \ldots, m_1$ are integers > 0.

By using the operation of multiplication by an ordinal $< \omega^\omega$: $A \to A.\alpha$, we could have replaced in the definition of $\mathcal{C}_{\mathbf{k+1}}$ the integers $n_1, n_2, \ldots, n_j$ by some non null ordinals $\nu_1, \nu_2, \ldots, \nu_j < \omega^\omega$. In such a way we generate many more Wadge degrees but the lower bounds for the lengths of the hierarchies remain unchanged.

As an example consider first the case of $\mathcal{C}_\mathbf{1}$ defined from the class $\mathcal{C}_\mathbf{0}$. Call $\mathcal{C}'_\mathbf{1}$ the family containing all conciliating sets in the following form and their complements:

$$(A_j)^\sim.\nu_j + (A_{j-1})^\sim.\nu_{j-1} + \ldots + (A_1)^\sim.\nu_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}_\mathbf{0}$, and $\nu_1, \nu_2, \ldots, \nu_j$ are some non null ordinals $< \omega^\omega$.

We used in the previous proof the Cantor normal form of an ordinal. It was in fact the Cantor normal form of base ω, but there exist some extensions: in particular every non null ordinal α has a Cantor normal form of base ω^ω, i.e. α may be written in the form [Sie65]:

$$\alpha = (\omega^\omega)^{\delta_j}.\nu_j + (\omega^\omega)^{\delta_{j-1}}.\nu_{j-1} + \ldots + (\omega^\omega)^{\delta_1}.\nu_1$$

where $j > 0$ is an integer, $\delta_j > \delta_{j-1} > \ldots > \delta_1$ are ordinals and $\nu_j, \nu_{j-1}, \ldots, \nu_1$ are non null ordinals $< \omega^\omega$.

Remark now that:

$$\omega^{\omega^\omega} = (\omega^\omega)^{\omega^\omega}$$

This follows from properties of arithmetical operations over ordinals. Indeed it holds that:

$$\omega.\omega^\omega = \omega^{1+\omega} = \omega^\omega$$

and then we can infer that:

$$\omega^{\omega^\omega} = \omega^{(\omega.\omega^\omega)} = (\omega^\omega)^{\omega^\omega}$$

Then the above normal form of base ω^ω describes an ordinal $\alpha < \omega^{\omega^\omega}$ iff every ordinal δ_i is $< \omega^\omega$.

Remark that for $A \in \mathcal{C}'_\mathbf{1}$, $d_c(A)$ is in the following form:

$$d_c(A) = \omega_1^{\alpha_j}.\nu_j + \omega_1^{\alpha_{j-1}}.\nu_{j-1} + \ldots + \omega_1^{\alpha_1}.\nu_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \mid A \in \mathcal{C}_\mathbf{0}\} \cup \{0\}$ and $\nu_j, \nu_{j-1}, \ldots, \nu_1$ are non null ordinals $< \omega^\omega$.

Consider now a non null ordinal $\alpha < \omega^{\omega^\omega} = \omega(3)$ written in the Cantor normal form of base ω^ω:

$$\alpha = (\omega^\omega)^{\delta_j}.\nu_j + (\omega^\omega)^{\delta_{j-1}}.\nu_{j-1} + \ldots + (\omega^\omega)^{\delta_1}.\nu_1$$

where $j > 0$ is an integer, $\delta_j > \delta_{j-1} > \ldots > \delta_1$ are ordinals $< \omega^\omega$ and $\nu_j, \nu_{j-1}, \ldots, \nu_1$ are non null ordinals $< \omega^\omega$.

Now it is easy to see that there exists a strictly increasing isomorphism:

$$\phi'_1 : \quad \{d_c(A) \ / \ A \in \mathcal{C}'_{\mathbf{1}}\} \longrightarrow \omega^{\omega^\omega} - \{0\}$$

$$\omega_1^{\alpha_j}.\nu_j + \omega_1^{\alpha_{j-1}}.\nu_{j-1} + \ldots + \omega_1^{\alpha_1}.\nu_1 \longrightarrow (\omega^\omega)^{\bar{\phi}_0(\alpha_j)}.\nu_j + (\omega^\omega)^{\bar{\phi}_0(\alpha_{j-1})}.\nu_{j-1} + \ldots + (\omega^\omega)^{\bar{\phi}_0(\alpha_1)}.\nu_1$$

where $j > 0$ is an integer, $\alpha_j > \alpha_{j-1} > \ldots > \alpha_1$ are in $\{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{0}}\} \cup \{0\}$ and $\nu_j, \nu_{j-1}, \ldots, \nu_1$ are non null ordinals $< \omega^\omega$. And where $\bar{\phi}_0$ is the strictly increasing isomorphism:

$$\bar{\phi}_0 : \quad \{0\} \cup \{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{0}}\} \longrightarrow \omega^\omega$$

defined in section 6.

Hence the length of the conciliating hierarchy of infinitary languages in $\mathcal{C}'_{\mathbf{1}}$ and also in $(1 - ICL_{\leq\omega}) \cap (Co - 1 - ICL_{\leq\omega})$ is greater than ω^{ω^ω} but we cannot get a better result.

The case of $\mathcal{C}_{\mathbf{k}}$ for $k \geq 2$ is very similar. We first remark that for each integer $n \geq 2$, it holds that $\omega.\omega(n) = \omega(n)$, and then:

$$\omega(n+1) = \omega^{\omega(n)} = \omega^{\omega.\omega(n)} = (\omega^\omega)^{\omega(n)}$$

Hence every ordinal $\alpha < \omega(n+1)$ admits a Cantor normal form of base ω^ω:

$$\alpha = (\omega^\omega)^{\delta_j}.\nu_j + (\omega^\omega)^{\delta_{j-1}}.\nu_{j-1} + \ldots + (\omega^\omega)^{\delta_1}.\nu_1$$

where $j > 0$ is an integer, $\delta_j > \delta_{j-1} > \ldots > \delta_1$ are ordinals $< \omega(n)$ and $\nu_j, \nu_{j-1}, \ldots, \nu_1$ are non null ordinals $< \omega^\omega$.

The proof is now similar to the case $k = 1$ but with a sligth modification: For $k \geq 1$ we define $\mathcal{C}'_{\mathbf{k+1}}$ as the family containing all conciliating sets in the following form and their complements:

$$(A_j)^{\sim}.\nu_j + (A_{j-1})^{\sim}.\nu_{j-1} + \ldots + (A_1)^{\sim}.\nu_1$$

where j is an integer ≥ 1, for each i, $1 \leq i \leq j$, $A_i \in \mathcal{C}'_{\mathbf{k}}$, and $\nu_1, \nu_2, \ldots, \nu_j$ are some non null ordinals $< \omega^\omega$.

But for $k \geq 1$ there exist in $\mathcal{C}'_{\mathbf{k}}$ some conciliating sets A which degrees are in the form $\alpha + n$, with α a limit ordinal of **cofinality** ω, and n an integer ≥ 0. Hence for these sets:

$$d_c(A^{\sim}) = (\omega_1)^{d_c(A)+1}$$

by Theorem 5.21. Nevertheless the order type of

$$\{d_c(A^{\sim}) \ / \ A \in \mathcal{C}'_{\mathbf{k}}\}$$

remains unchanged and is equal to the order type of $\{d_c(A) \ / \ A \in \mathcal{C}_{\mathbf{k}}\}$. Further details are left to the reader. □

8 Concluding remarks and further work

We proved in [Fin99a] that the class CFL_ω exhausts the hierarchy of Borel sets of finite rank.
We have proved above that the length of the Wadge hierarchy of ω-CFL is greater than ε_0.

On the other hand, deterministic ω-CFL are all boolean combinations of $\mathbf{\Sigma^0_2}$-sets therefore they are $(\mathbf{\Sigma^0_3} \cap \mathbf{\Pi^0_3})$-sets. And the Wadge hierarchy of deterministic ω-CFL has length $\omega^{(\omega^2)}$. This hierarchy is studied by J. Duparc in [Dup99] using methods of descriptive set theory and game theory. We shall present in future papers a study of this hierarchy which is analogous to Wagner's study of the Wadge hierarchy of ω-regular languages, [Fin99b].

Thus our results show that, with regard to their topological complexity, non deterministic pushdown automata have a much stronger expressive power than deterministic pushdown automata, when reading ω-words with a Büchi or Muller acceptance condition.

And this is in big contrast with the case of finite automata, because deterministic and non deterministic Muller automata have exactly the same expressive power and define boolean combinations of $\mathbf{\Sigma^0_2}$-sets.

Further, it remains to determine the exact length of the Wadge hierarchy of ω-CFL (and of the other hierarchies we have studied here) and all the degrees of ω-CFL. And, although the Wadge hierarchy of ω-CFL is not effective, it seems possible, as stated in [DFR99], to find some subclass of CFL_ω which would strictly contain the class of deterministic ω-CFL but would have an effective Wadge hierarchy.

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